Real Analysis in Computer Science: A collection of Open Problems

Yuval Filmus, Hamed Hatami, Steven Heilman, Elchanan Mossel, Ryan O'Donnell, Sushant Sachdeva, Andrew Wan, and Karl Wimmer*

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Abstract

We list a collection of open problems in real analysis in computer science, which complements, updates and extends a previous list curated by Ryan O'Donnell (2012). The object of study in these problems are boolean functions $f\colon \{0,1\}^n \to \{0,1\}$, and their analytic and combinatorial properties. Many of the questions originate from in theoretical computer science or the theory of voting. The formulation of many of the questions has a strong combinatorial and analytical flavor including the use of the discrete Fourier expansion.

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1 Introduction

Fourier analysis of boolean functions is an active area of research which brings together mathematical analysis, theoretical computer science and the theory of voting. During the 2012 Simons symposium on the area, O'Donnell [O'D12] curated a list of open problems in the area. The current collection curated by the authors while attending the special semester on *real analysis in computer science* at the Simons Institute during the fall of 2013, includes additional open problems as well as update on the status of some of the open problems presented in [O'D12].

Common notations and definitions are listed in Section 2, which is followed by several sections listing open problems organized thematically. For general background we refer to reader to the recent book [O'D14].

2 Notations and Definitions

We use the notation $[n] = \{1, \ldots, n\}$. We denote the cardinality of a set S by $|S|$. The finite field with p points is denoted $\mathbb{F}_p$. The *support* of a function f, denoted $\mathrm{Supp}(f)$, is the set of points in the domain on which f is non-zero. The notation $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2.

2.1 Analysis on the boolean cube

Boolean cube. The boolean cube $\{0,1\}^n$ is endowed with the uniform measure. The inner product between two functions $f, g \colon \{0,1\}^n \to \mathbb{R}$ is given by $\langle f, g \rangle = \mathbb{E}_{x \in \{0,1\}^n} f(x)g(x)$, and the L^2 norm is given by $\|f\| = \|f\|_2 = \sqrt{\langle f, f \rangle}$. Other norms are given by $\|f\|_p = (\mathbb{E}_{x \in \{0,1\}^n} |f(x)|^p)^{1/p}$.

Fourier expansion. A *boolean function* on n inputs is a function $f \colon \{0,1\}^n \to \{0,1\}$. It is often prudent to consider more generally real-valued functions $f \colon \{0,1\}^n \to \mathbb{R}$. The *Fourier basis* consists of the functions $\chi_S(x_1, \ldots, x_n) = (-1)^{\prod_{i \in S} x_i}$. (Sometimes we use the notation W_S instead of χ_S.) The Fourier basis is an orthonormal basis of the space of all functions from $\{0,1\}^n$ to $\mathbb{R}$, and so every function $f \colon \{0,1\}^n \to \mathbb{R}$ has a unique *Fourier expansion*

$$f = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S, \qquad \text{where } \hat{f}(S) = \langle f, \chi_S \rangle = \mathop{\mathbb{E}}_{x \in \{0,1\}^n} f(x) \chi_S(x).$$

Parseval's identity states that

$$\mathop{\mathbb{E}}_{x \in \{0,1\}^n} [f(x)^2] = \sum_{S \subseteq [n]} \hat{f}(S)^2.$$

In particular,

$$\mathop{\mathbb{V}}_{x \in \{0,1\}^n} [f] = \sum_{S \neq \emptyset} \hat{f}(S)^2.$$

Functions given by polynomials. Every boolean function can be expressed uniquely as a multilinear polynomial. A *polynomial of degree d* is a boolean function given by a multilinear polynomial of degree d.

Influence. Let $f\colon \{0,1\}^n \to \{0,1\}$ be a boolean function. The *influence* of the ith variable is

$$\mathrm{Inf}_i(f) := \Pr_{x\in\{0,1\}^n}[f(x) \neq f(\Delta_i x)],$$

where $\Delta_i x$ is the vector differing from x in the ith coordinate only. In the literature sometimes the alternative notation $I_i(f)$ is used. Direct calculation gives an equivalent spectral definition:

$$\mathrm{Inf}_i(f) = 4\sum_{i\in S} \hat{f}(S)^2,$$

and this definition makes sense for arbitrary real-valued functions $f\colon \{0,1\}^n \to \mathbb{R}$. Going back full circle, we discover the general spatial definition

$$\mathrm{Inf}_i(f) := \mathop{\mathbb{E}}_{x\in\{0,1\}^n}[(f(x) - f(\Delta_i x))^2].$$

The *total influence* (or *average sensitivity*) of a function $f\colon \{0,1\}^n \to \{0,1\}$ is the sum of the individual influences:

$$\mathrm{Inf}(f) := \sum_{i=1}^n \mathrm{Inf}_i(f).$$

The matching spectral expression is

$$\mathrm{Inf}(f) = \sum_{S\subseteq[n]} |S|\hat{f}(S)^2.$$

Poincaré's inequality $\mathrm{Inf}(f) \geq 4\,\mathbb{V}(f)$ easily follows.

Some authors prefer defining the influence without the constant factor 4. The spatial definition then has to be adjusted accordingly.

Smoothing. For $\rho \in [-1,1]$, the *smoothing operator* (also known as the *noise operator* or the *Ornstein–Uhlenbeck* operator) T_ρ, operating on functions $f\colon \{0,1\}^n \to \mathbb{R}$, is given spectrally by

$$(T_\rho f)(x) := \sum_{S\subseteq[n]} \rho^{|S|}\hat{f}(S)\chi_S(x).$$

The spatial definition is slightly more complicated. For $x \in \{0,1\}^n$, let $x_\rho \in \{0,1\}^n$ be obtained by independently perturbing each coordinate so that the correlation coefficient between x_i and $x_{\rho,i}$ is ρ. Then

$$(T_\rho f)(x) = \mathop{\mathbb{E}}_{x_\rho} f(x_\rho).$$

The noise operators form a semigroup: $T_\alpha T_\beta = T_{\alpha\beta}$. For this reason, sometimes the parametrization $T'_\rho = T_{e^\rho}$ is used: it makes $T'_\alpha T'_\beta = T'_{\alpha+\beta}$.

Replacing $\{0,1\}$ with $\{-1,1\}$. So far we have considered functions with domain $\{0,1\}^n$. Sometimes it is nicer to consider the domain $\{-1,1\}^n$; here $+1$ corresponds to 0 and -1 corresponds to 1. The Fourier characters then become $\chi_S(x_1,\ldots,x_n) = \prod_{i\in S} x_i$. All other definitions are the same. The spatial definition of the smoothing operator becomes simpler: x_ρ is obtained by independently perturbing each coordinate of x so that $\mathbb{E}[x_i x_{\rho,i}] = \rho$.

Similarly, sometimes we want to consider boolean functions whose *range* is $\{-1,1\}$, using the same correspondence. The spatial definition of influence has to be changed slightly:

$$\mathrm{Inf}_i(f) := 4 \Pr_{x\in\{0,1\}^n}[f(x) \neq f(\Delta_i x)].$$

Some authors prefer keeping the original spatial definition (without the factor 4), and in that case the spectral definition becomes simply

$$\mathrm{Inf}'_i(f) = \sum_{i\in S} \hat{f}(S)^2.$$

Biased measures. Our definitions used the *uniform* inner product $\langle f,g\rangle = \mathbb{E}_{x\in\{0,1\}^n} f(x)g(x)$. It is sometimes advantageous to consider a skewed distribution on $\{0,1\}^n$ instead. For $0 \le p \le 1$, the *p-biased measure*, denoted μ_p or μ_p^n, is given by $\mu_p(x) = p^{|x|}(1-p)^{n-|x|}$. All notions considered so far can be extended to the biased case; see O'Donnell's book [O'D14].

2.2 Analysis on $[m]^n$

The Fourier basis is just the basis of characters of the abelian group $\mathbb{Z}_2^n$. We can similarly define the Fourier expansion of a function $f\colon [m]^n \to \mathbb{R}$ by taking the characters of $\mathbb{Z}_m^n$. The Fourier basis is $\chi_S(x_1,\ldots,x_n) = \omega^{\sum_{i=1}^n S_i x_i}$, where $S \in [m]^n$ and ω is a primitive mth root of unity.

There are several different extensions of the notion of influence; since the notion doesn't come up in the problems, we do not define any of them here.

The noise operator is given by

$$T_\rho f := \sum_{S\in[m]^n} \rho^{|S|} \hat{f}(S) \chi_S.$$

The corresponding spatial definition is similar to the previous one: given $\rho \in [-1,1]$ and a vector $x \in [m]^n$, define a vector $x_\rho \in [m]^n$, define $x_{\rho,i} = x_i$ with probability $(1+\rho)/2$, and choose $x_{\rho,i}$ randomly from the values different from x_i with probability $(1-\rho)/2$. Then

$$T_\rho f = \mathbb{E}_{x_\rho}[f(x_\rho)].$$

2.3 Analysis on $\mathbb{R}^n$

The invariance principle allows us to reason about functions on the boolean cube using function on $\mathbb{R}^n$. All the notions we have considered can be extended to this case as well, though we will only describe how to extend the Ornstein–Uhlenbeck operator. For a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and $\rho \in [-1,1]$, define

$$(T_\rho f)(x) := \mathbb{E}_{y\sim\mathcal{N}(0_n,I_n)}[f(\rho x + \sqrt{1-\rho^2}y)].$$

Here $\mathcal{N}(0_n, I_n)$ is the multivariate normal distribution of dimension n. In other words, each coordinate of y is an independent normally distributed random variable with zero mean and unit variance.

2.4 Computational complexity

Circuits. Boolean circuits on n bits are boolean functions $\{0,1\}^n \to \{0,1\}$ specified by labelled DAGs (directed acyclic graphs) with a unique sink (node with out-degree 0) of the following form:

- Each leaf is labelled with one of $0, 1, x_1, \ldots, x_n$.
- Each internal node is labelled by one of NOT, AND, OR; the label NOT can only be used if the in-degree is 1.

We define inductively the value of a node in the circuit in the obvious way:

- A leaf has the value of the constant or variable which it is labelled by.
- A node labelled NOT has the value 1 if its child has the value 0, and it has the value 0 if its child has the value 1.
- A node labelled AND has the 1 if all its children have value 1, and the value 0 otherwise.
- A node labelled OR has the 0 if all its children have value 0, and the value 1 otherwise.

The function computed by the circuit is the value of the unique sink. Nodes are usually known as *gates*, and the sink is known as the *root*. If there are no NOT gates, the circuit is *monotone*, and one can check that it computes a monotone function (a *monotone function* is one satisfying $f(x) \leq f(y)$ for $x \leq y$). If the out-degree of every node is at most 1, the circuit is a *formula*. The *size* of the circuit is the number of nodes. The *depth* of the circuit is the maximal length of a root-to-leaf path (here *length* is the number of edges; some authors prefer to count vertices). What we have described is circuits with *unbounded fan-in* (the *fan-in* of a node is its in-degree). Some circuit classes are defined with a restriction on the fan-in, but we will not need them here.

Circuit classes. The class AC^0 consists of sequences of boolean functions $f_n \colon \{0,1\}^n \to \{0,1\}$ such that f_n can be computed by a circuit of size $n^{O(1)}$ and depth $O(1)$. In the computation complexity literature, this class is known more accurately as *non-uniform* AC^0 or AC^0/poly.

The class TC^0 is defined similarly, in terms of *threshold circuits*, which are circuits with the addition of a new type of gate, a *threshold* gate: this gate computes the indicator function of $\sum_i \alpha_i x_i \geq T$, where x_i are the inputs and α_i, T are parameters. If all α_i are non-negative then the threshold gate is *monotone*. A threshold circuit is *monotone* if there are no NOT gates and all threshold gates are monotone.

Decision trees. Decision trees are a different model for defining boolean functions. A decision tree on n bits is a tree in which each internal node has a exactly two children, a 0-child and a 1-child. Each node is labelled by a variable, and each leaf is labelled by a binary value. The decision tree gives instructions for computing a function: Starting with the root, for each internal node, computation proceeds to either the 0-child or the 1-child according to the value of the variable that labels the node. The computation eventually reaches a leaf, which is the value computed by the decision tree. The *depth* of the decision tree is the length of the maximum root-to-leaf path.

A *randomized decision tree* is a probability distribution over decision trees. The *worst-case depth* of a randomized decision tree is the maximum over all inputs of the expected length of the root-to-leaf computed by a random decision tree chosen according to the given distribution.

3 Boolean functions

~~KKL inequality under K-wise independence~~

Statement: Suppose $f\colon \{0,1\}^n \to \{0,1\}$ satisfies $\widehat{f}(S) = 0$ for every $S \subseteq [n]$ with $|S| \leq K_n$, where $\lim_{n\to\infty} K_n = \infty$. Is it true that $\max_{i\in[n]} \mathrm{Inf}_i(f) = \omega(\frac{\log n}{n})$? In other words, is it true that $\max_{i\in[n]} \mathrm{Inf}_i(f) \geq \frac{\omega_n \log n}{n}$ for some ω_n with $\lim_{n\to\infty} \omega_n = \infty$?
Source: Hamed Hatami.
Remarks: A counter example based on the tribes function and good codes is provided in [HMO14].

~~Harper's inequality under K-wise independence~~

Statement: Suppose $f\colon \{0,1\}^n \to \{0,1\}$ satisfies $\widehat{f}(S) = 0$ for every $S \subseteq \{1,\dots,n\}$ with $1 \leq |S| \leq K_n$ where $\lim_{n\to\infty} K_n = \infty$. Let $\mathbb{E}$ be expectation with respect to the uniform probability measure on $\{0,1\}^n$. Is it true that $\sum_{i=1}^n \mathrm{Inf}_i(f) \geq \omega_n \mathbb{E}[f] \log \frac{1}{\mathbb{E}[f]}$ for some ω_n with $\lim_{n\to\infty} \omega_n = \infty$?
Source: Gil Kalai

A counter example based on the tribes function and good codes is provided in [HMO14].

Largest influence and the degree

Statement: What is the smallest possible value for $\max_{i\in[n]} \mathrm{Inf}_i(f)$ for a balanced (i.e. $\mathbb{E}[f] = 1/2$) boolean function $f : \{0,1\}^n \to \{0,1\}$ with $\widehat{f}(S) = 0$ for all $|S| > d$, $S \subset \{1,\dots,n\}$?
Source: Hamed Hatami
Remarks:

- By a result of Nisan and Smolensky (See [BdW02]) f has a decision tree of depth $O(d^4)$, and then a result of O'Donnell, Saks, Schramm, and Servedio [OSSS05] implies $\max_{i\in[n]} \mathrm{Inf}_i(f) \geq \Omega(d^{-4})$.

Small influences and influential tribes

Statement: For every $c > 0$ and $\epsilon > 0$ there exists $b > 0$ such that, if an increasing $f\colon \{0,1\}^n \to \{0,1\}$ and for all $i \in \{1,\dots,n\}$, if $\mathrm{Inf}_i(f) \leq c\log(n)/n$, then there exists a small "tribe", i.e., a set J of variables with $|J| \leq b\log(n)$, such that either $\mathbb{E}[f(x)|x_J = \vec{1}] \geq 1 - o(1)$ or $\mathbb{E}[f(x)|x_J = \vec{0}] \leq o(1)$?
Source: Friedgut [Fri04].
Remarks:

- One can also formulate a general form of the conjecture without the increasing assumption (either $\max_{y\in\{0,1\}^J} \mathbb{E}[f(x)|x_J = y] \geq 1 - o(1)$ or $\min_{y\in\{0,1\}^J} \mathbb{E}[f|x_J = y] \leq o(1)$.)

Influential tribes for continuous cube

Statement: Let $f\colon [0,1]^n \to \{0,1\}$ be an increasing measurable function. There exists a set $J \subseteq \{1,\dots,n\}$ with $|J| = o(n)$ such that either $\mathbb{E}[f(x)|x_J = \vec{1}] \geq 1 - o(1)$ or $\mathbb{E}[f(x)|x_J = \vec{0}] \leq o(1)$.
Source: Friedgut [Fri04]
Remarks:

- It was wrongfully claimed in [BKK+92] that it is always possible to achieve $\mathbb{E}[f(x)|x_J = \vec{1}] \geq 1 - o(1)$ provided $\mathbb{E}[f] = \Omega(1)$. However defining $f(x_1, \dots, x_n) = 1$ if and only if $x_i > \frac{1}{n}$ for all $1 \leq i \leq n$ refutes this.

Stability of the edge-isoperimetric inequality

Statement: For every $K > 0$, there are positive real numbers $K', \delta > 0$ depending on K such that the following assertion holds. If a monotone boolean function $f\colon \{0,1\}^n \to \{0,1\}$ with $\mathbb{E}[f] \leq \frac{1}{2}$ satisfies $\sum_{i=1}^{n} \mathrm{Inf}_i(f) \leq K\mathbb{E}[f]\log(1/\mathbb{E}[f])$, then there is a set J of at most $-K' \log \mathbb{E}[f]$ coordinates such that $\mathbb{E}[f(x)|x_J = \vec{1}] \geq (1+\delta)\mathbb{E}[f]$.
Source: [KK07].
Remarks:

- The case of $\mathbb{E}[f] = \Omega(1)$ follows from Friedgut's junta theorem [Fri98]. See also [FS07] for some relevant results and discussions.

Expected vs. critical threshold

Statement: For a graph H, let $p_c(H)$ denote the critical probability defined as $\Pr[G(n, p_c) \supseteq H] = \frac{1}{2}$, and let $P_E(H)$ be the expected threshold defined as the least p such that for all $H' \subseteq H$, we have

$$\mathbb{E}[\text{number of copies } H' \text{ in } G(n,p)] \geq \frac{1}{2}.$$

Given $\epsilon > 0$ is there a K such that any H with $\epsilon < p_c(H) < 1 - \epsilon$ satisfies $p_c(H) < K p_E(H)$?
Source: [KK07].
Remarks:

- For arbitrary H (e.g. a hamiltonian cycle) it is possible that $p_c(H) \gtrsim (\log n) p_E(H)$. However it is also conjectured by Kahn and Kalai [KK07] that $p_c(H) \lesssim (\log n) p_E(H)$ holds for every graph H.

Monotone DNFs and pseudo-juntas

Statement: Recall that a DNF is an OR of ANDs, and a DNF is called monotone if it only contains un-negated boolean variables. Let $\epsilon > 0$ and let $f\colon \{0,1\}^n \to \{0,1\}$ be defined by a monotone DNF. Let $p = p(n)$ and denote by μ_p^n the p-biased product distribution on $\{0,1\}^n$. If for a random assignment $x \sim \mu_p^n$, the expected number of variables that belong to at least one satisfied clause is K. Is there a DNF g with clauses of size at most $O_{\epsilon,K}(1)$ such that $\Pr_{x\sim\mu_p^n}[f(x) \neq g(x)] \leq \epsilon$?
Source: [Hat12].
Remarks:

- The main result of [Hat12] implies that this is equivalent to [Fri99, Conjecture 1.5].

The daisies problem

Statement: For $x \in \{0,1\}^n$ and $S \subseteq \{1, \dots, n\}$, let $\Delta_S(x) \in \{0,1\}^n$ be the vector that is equal to x in the coordinates in $\{1, \dots, n\} \setminus S$ and different from x in the coordinates in S. For every $\epsilon > 0$, and integers $0 \le s \le t$, if n is sufficiently large, and $A \subseteq \{0,1\}^n$ satisfies $\mathbb{E}[A] > \epsilon$, then there exists $T \subseteq \{1,\dots,n\}$ of size t, and $x \in \{0,1\}^n$ such that

$$\Delta_S(x) \in A \quad \text{for all } S \subseteq T \text{ with } |S| = s.$$

Source: [BLM11].
Remarks:

- This is open even in the case of $t = 4$ and $s = 2$.

Extremal sets in the hypercube

Statement: Consider integers $1 \le d \le n$ and a family $\mathcal{F}$ of subsets $F \subseteq \{0,1\}^d$. We say that $S \subseteq \{0,1\}^n$ is $\mathcal{F}$-free if every embedding $\iota \colon \{0,1\}^d \to \{0,1\}^n$ satisfies $\iota(F) \not\subseteq S$ for all $F \in \mathcal{F}$. Is it true that among all $\mathcal{F}$-free sets $S \subseteq \{0,1\}^n$, there is a symmetric largest set?
Source: [JT10].
Remarks:

- A positive answer would imply a positive answer to the daisies problem [BLM11] mentioned above.

Testing correlation with cubic polynomials

Statement: Does there exist a test which queries a given $f \colon \mathbb{F}_2^n \to \mathbb{F}_2$ on a constant number of points, and distinguishes whether f has noticeable or negligible correlation with cubic polynomials $p \colon \mathbb{F}_2^n \to \mathbb{F}_2$?

More precisely, is there a universal constant k, and a protocol (an adaptive procedure) such that given a function $f \colon \mathbb{F}_2^n \to \mathbb{F}_2$, makes k queries to f, and for every $\epsilon > 0$,

- Accepts f with probability $\alpha(\epsilon) > \frac{1}{2}$ if $\max_{p:\deg(p)\le 3} \mathbb{E}(-1)^{f(x)+p(x)} \ge \epsilon$;
- Rejects f with probability $\alpha(\epsilon) > \frac{1}{2}$ if $\max_{p:\deg(p)\le 3} \mathbb{E}(-1)^{f(x)+p(x)} < \delta$, where $\delta > 0$ can depend on ϵ.

Source: [HL11].
Remarks:

- The Gowers U^2 and U^3 norms provide such tests for correlation with linear and quadratic polynomials, respectively. However the example of the symmetric polynomial of degree 4 in [LMS11, GT09a] shows that the U^4 norm does not provide such a test for correlation with cubic polynomials.

Small support implies large biased subcubes

Statement: Suppose $f \colon \{0,1\}^n \to \mathbb{R}$ satisfy $|\mathrm{Supp}(f)| = m$. Show that there is a $S \subset \{1,\dots,n\}$ of size $O(\log m)$ such that $|\mathbb{E}[f(x)|x_S = y]| > \|f\|_1/\mathrm{poly}(m)$ for some fixed $y \in \{0,1\}^S$.
Source: Shachar Lovett.
Remarks:

- The case of boolean function $f\colon \{0,1\}^n \to \{-1,0,1\}$ follows from the fact that one can find a set S and an y such that there exists only one $x \in \mathrm{Supp}(f)$ with $x_S = y$. The statement is also true for symmetric functions [Shachar Lovett, personal communication].

Corners in quasi-random groups

Statement: Let $\delta > 0$. Is it true that for sufficiently large q and $G := \mathrm{PSL}_2(q)$, the following holds? For every subset $A \subseteq G \times G$ with density $\frac{|A|}{|G|^2} \geq |G|^{2-\delta}$, there exists $x, y, z \in G$ with $z \neq 1$ such that $(x, y), (xz, y), (x, zy) \in A$.
Source: Hamed Hatami.
Remarks:

- Bounds for similar problems are proven in [Tao13, Gow08, BT14, LM07, Shk06]. Proving the conjecture for any group G would be as interesting. The motivation for this problem comes from communication complexity. Consider the function $f\colon G \times G \times G \to \{0,1\}$ defined as $f(x,y,z) = 1$ if and only if $xyz = 1$. Using a simple hashing argument, one can see that f has a $O(1)$ randomized communication protocol in the NOF model. If the above conjecture is true, then one needs $\Omega(\log |G|)$ bits of communication in the deterministic NOF model. To see this let A' be a cylinder-intersection (i.e. $A' = (B_1 \times C_1 \times G) \cap (B_2 \times G \times D_2) \cap (G \times C_3 \times D_3)$) containing only elements (x, y, z) with $f(x, y, z) = 1$. Let $A = \{(x, y) : (x, y, y^{-1}x^{-1}) \in A'\}$. If $|A'| = |A| \geq |G|^{2-\delta}$, then the conjecture would imply the existence of $(x, y, y^{-1}x^{-1}), (xz, y, y^{-1}z^{-1}x^{-1}), (x, zy, y^{-1}z^{-1}x^{-1}) \in A'$. Since A' is a cylinder-intersection, this leads to $(x, y, y^{-1}z^{-1}x^{-1}) \in A'$ contradicting $z \neq 1$. Hence one needs at least $|G|^\delta$ monochromatic cylinder-intersections to cover all of $G \times G \times G$, and this provides the desired communication complexity lower-bound.

AC^0 and Fourier spectrum

Statement: Let $\epsilon > 0$ be a fixed real number and $c \geq 1$ be a fixed integer. Let $f\colon \{0,1\}^n \to \{0,1\}$ be a monotone property expressed by a depth-c circuit of size M. There is a set S of polynomial size in M (where the polynomial depends on c and ϵ) so that

$$\sum_{S \notin \mathcal{S}} \widehat{f}(S)^2 \leq \epsilon.$$

Source: [BKS99, Conjecture 7.2].

TC^0 and Fourier spectrum

Statement: In a threshold circuit each gate is a weighted majority function with an arbitrary number of inputs. In a monotone threshold circuit all the weights in every gate are non-negative. Let f be a boolean function given by a monotone threshold circuit of depth c and size M. Then

$$\sum_{S \neq \emptyset} \frac{\widehat{f}(S)^2}{|S|} \leq O\left((\log M)^{c-1}\right).$$

Source: [BKS99, Conjecture 7.3].

Graph properties and AC^0

Statement: There are no balanced graph properties in AC^0. Here a *graph property* is a subset of graphs which is invariant under permutation of the vertices, and *balanced* means that a $G(n, 1/2)$ random graph satisfies the property with probability tending to $1/2$.
Source: Ben Rossman.
Remarks:

- It follows from Rossman's work that containing a clique of size $\sim 2\log n$ is such a property [Hamed Hatami, Personal communication].

Almost-resiliency of monotone functions

Statement: Let $\mathcal{C}_n$ be the class of boolean functions $f\colon \{0,1\}^n \to \{0,1\}$ that are monotone and depend on all n variables. For each $n \geq 3$, let $d(n)$ be the largest integer d such that for some $f \in \mathcal{C}_n$, $\widehat{f}(S) = 0$ for all S where $2 \leq |S| \leq d$. What is the behavior of $d(n)$ as $n \to \infty$?
Source: Karl Wimmer.
Remarks:

- The best bounds known are $2 \leq d(n) \leq \Omega(\sqrt{n})$. Because f is monotone and depends on all n variables, $\widehat{f}(\{i\}) \neq 0$ for $1 \leq i \leq n$. If the monotonicity condition is dropped and replaced with $\widehat{f}(\{i\}) \neq 0$ for $1 \leq i \leq n$, the best bounds known are $5 \leq d(n) \leq n - \Omega(1)$.

L_0 sparsity of spectrum

Statement: Let $f\colon \{0,1\}^n \to \{0,1\}$. Suppose that $\hat{f}(\{i\}) \neq 0$ for all $1 \leq i \leq n$. Show that the number of non-zero coefficients of f is at least $\Omega(n^2)$.
Source: John Wright (1st workshop at the Real Analysis special semester)
Remarks:

- The address function $f\colon \{0,1\}^{r+2^r} \to \{0,1\}$ defined by

$$f(x_0, \ldots, x_{2^r-1}, y_1, \ldots, y_r) = x_{y_1,\ldots,r_r}.$$

  is a function that satisfies $\hat{f}(j) \neq 0$ for all $0 \leq j < 2^r$ but has $O(2^{2r})$ non-zero coefficients.

Size of the junta in Bourgain's theorem

Statement: Let $f\colon \{0,1\}^n \to \{0,1\}$ be a boolean function and $\epsilon = \|\sum_{|S|>k} \hat{f}(S)\chi_S\|^2$. Show that f is $O(\sqrt{k}\epsilon)$-close to some $\frac{2^{O(k)}}{\epsilon}$-junta.
Source: [KKO].
Remarks:

- [KKO] get a junta of size $\frac{2^{O(k)}}{\epsilon^4}$.

Decision-tree complexity of recursive 3-majority

Statement: Let $\mathrm{Maj}\colon \{0,1\}^3 \to \{0,1\}$ denote the majority function on 3 input bits. Let $x = (x_1, \ldots, x_{3^h}) \in \{0,1\}^{3^h}$. Let $\mathrm{Maj}_0\colon \{0,1\} \to \{0,1\}$ be the identity function. For every $h \geq 0$, $h \in \mathbb{Z}$, define the recursive 3-Majority function so that

$$\begin{aligned}&\mathrm{Maj}_h(x)\\ &:= \mathrm{Maj}\left(\mathrm{Maj}_{h-1}(x_1, \ldots, x_{3^{h-1}}), \mathrm{Maj}_{h-1}(x_{1+3^{h-1}}, \ldots, x_{2\cdot 3^{h-1}}), \mathrm{Maj}_{h-1}(x_{1+2\cdot 3^{h-1}}, \ldots, x_{3^h})\right).\end{aligned}$$

We're looking for a randomized decision tree which minimizes the worst-case (over inputs) cost of computing recursive 3-majority. The best bounds we currently know are 2.5^n (lower bound) and $\approx 2.65^n$ (upper bound, 2011).
Source: [MNS+13]
Remarks:

- The following algorithm *isn't* optimal: query two children (recusively), and query the third one only if the first two disagree. This gives $(8/3)^n$ since worst-case inputs have 2-vs.-1 on every internal node. Can be improved by looking at two levels at once. What's the best that can be achieved?

Low-degree polynomials and distance from indicators

Statement: Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d such that $\mathrm{Var}[p] \geq 1$. It is known (see [KO12]) that for some constant c:

$$\mathbb{E}[(|p|-1)^2] \geq \frac{c}{\sqrt{d}},$$

where the expectation is with respect to the standard Gaussian distribution. What is the correct value of c? It is conjectured that c is of the form $c_m(1-o_d(1))$, where c_m is the constant obtained by setting $p = \sum_{|S|\leq d} \hat{f}(S)\cdot H_S$, where the H_S are the n-variate Hermite polynomials. The same question may be asked for polynomials over $\{-1,1\}^n$, with the requirement that all influences are at most some τ; then the lower bound is allowed an additive factor depending on τ.
Source: Guy Kindler, personal communication.

Correlation Bounds for Polynomials

Statement: Find an explicit (i.e., in NP) function $f\colon \mathbb{F}_2^n \to \mathbb{F}_2$ such that we have the correlation bound $|\mathbb{E}[(-1)^{\langle f(\boldsymbol{x}), p(\boldsymbol{x})\rangle}]| \leq 1/n$ for every $\mathbb{F}_2$-polynomial $p\colon \mathbb{F}_2^n \to \mathbb{F}_2$ of degree at most $\log_2 n$.
Source: Folklore dating back to [Raz87, Smo87].
Remarks:

- The problem appears to be open even with correlation bound $1/\sqrt{n}$ replacing $1/n$.
- Define the mod_3 function to be 1 if and only if the number of 1's in its input is congruent to 1 modulo 3. Smolensky [Smo87] showed that mod_3 has correlation at most 2/3 with every $\mathbb{F}_2$-polynomial of degree at most $c\sqrt{n}$ (where $c > 0$ is an absolute constant). For related bounds using his techniques, there seems to be a barrier to obtaining correlation $o(1/\sqrt{n})$.
- Babai, Nisan, and Szegedy [BNS92] implicitly showed a function in P which has correlation at most $\exp(-n^{\Theta(1)})$ with any $\mathbb{F}_2$-polynomial of degree at most $.99\log_2 n$; see also [VW08]. Bourgain [Bou05] (see also [GRS05]) showed a similar (slightly worse) result for the mod_3 function.

Tomaszewski's Conjecture

Statement: Let $a \in \mathbb{R}^n$ have $\|a\|_2 = 1$. Then $\Pr_{\boldsymbol{x}\sim\{-1,1\}^n}[|\langle a, \boldsymbol{x}\rangle| \leq 1] \geq 1/2$.
Source: Question attributed to Tomaszewski in [Guy86].
Remarks:

- The bound of 1/2 would be sharp in light of $a = (1/\sqrt{2}, 1/\sqrt{2})$.

- Holman and Kleitman [HK92] proved the lower bound 3/8. In fact they proved the inequality $\Pr_{\boldsymbol{x}\sim\{-1,1\}^n}[|\langle a,\boldsymbol{x}\rangle| < 1] \geq 3/8$ (assuming $a_i \neq \pm 1$ for all i), which is sharp in light of $a = (1/2, 1/2, 1/2, 1/2)$.

Bernoulli Conjecture

Statement: Let T be a finite collection of vectors in $\mathbb{R}^n$. Define $b(T) = \mathbb{E}_{\boldsymbol{x}\sim\{-1,1\}^n}[\max_{t\in T}\langle t,\boldsymbol{x}\rangle]$, and define $g(T)$ to be the same quantity except with $\boldsymbol{x} \sim \mathbb{R}^n$ Gaussian. Then there exists a finite collection of vectors T' such that $g(T') \leq O(b(T))$ and $\forall t \in T\ \exists t' \in T'\ \|t - t'\|_1 \leq O(b(T))$.
Source: [Tal94].
Remarks:

- The quantity $g(T)$ is well-understood in terms of the geometry of T, thanks to Talagrand's majorizing measures theorem.
- Talagrand offers a \$5000 prize for proving this, and a \$1000 prize for disproving it.
- The conjecture was prove by Bednorz and Latała [BL13].

Aaronson–Ambainis Conjecture

Statement: Let $f\colon \{-1,1\}^n \to [-1,1]$ have degree at most k. Then there exists $i \in [n]$ with $\mathbf{Inf}_i(f) \geq (\mathbb{V}[f]/k)^{O(1)}$.
Source: [Aar08, AA11a].
Remarks:

- True for $f\colon \{-1,1\}^n \to \{-1,1\}$; this follows from a result of O'Donnell, Schramm, Saks, and Servedio [OSSS05].
- The weaker lower bound $(\mathbb{V}[f]/2^k)^{O(1)}$ follows from a result of Dinur, Kindler, Friedgut, and O'Donnell [DFKO07].

Bhattacharyya–Grigorescu–Shapira Conjecture

Statement: Let $M \in \mathbb{F}_2^{m\times k}$ and $\sigma \in \{0,1\}^k$. Say that $f\colon \mathbb{F}_2^n \to \{0,1\}$ is (M,σ)-*free* if there does not exist $X = (x^{(1)},\dots,x^{(k)})$ (where each $x^{(j)} \in \mathbb{F}_2^n$ is a row vector) such that $MX = 0$ and $f(x^{(j)}) = \sigma_j$ for all $j \in [k]$. Now fix a (possibly infinite) collection $\{(M^1,\sigma^1),(M^2,\sigma^2),\cdots\}$ and consider the property $\mathcal{P}_n$ of functions $f\colon \mathbb{F}_2^n \to \{0,1\}$ that f is (M^i,σ^i)-free for all i. Then there is a one-sided error, constant-query property-testing algorithm for $\mathcal{P}_n$.
Source: [BGS10].
Remarks:

- The conjecture is motivated by a work of Kaufman and Sudan [KS08] which proposes as an open research problem the characterization of testability for linear-invariant properties of functions $f\colon \mathbb{F}_2^n \to \{0,1\}$. The properties defined in the conjecture are linear-invariant.
- Every property family $(\mathcal{P}_n)$ defined by $\{(M^1,\sigma^1),(M^2,\sigma^2),\cdots\}$-freeness is *subspace-hereditary*; i.e., closed under restriction to subspaces. The converse also "essentially" holds. [BGS10].

- For M of rank one, Green [Gre05b] showed that $(M, 1^k)$-freeness is testable. He conjectured this result extends to arbitrary M; this was confirmed by Král', Serra, and Vena [KSV12] and also Shapira [Sha09]. Austin [Sha09] subsequently conjectured that (M, σ)-freeness is testable for arbitrary σ; even this subcase is still open.
- The conjecture is known to hold when all M^i have rank one [BGS10]. Also, Bhattacharyya, Fischer, and Lovett [BFL12] have proved the conjecture in the setting of $\mathbb{F}_p$ for affine constraints $\{(M^1, \sigma^1), (M^2, \sigma^2), \dots\}$ of "Cauchy–Schwarz complexity" less than p.

Linear Coefficients versus Total Degree

Statement: Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Then $\sum_{i=1}^n \widehat{f}(i) \leq \sqrt{\deg(f)}$.
Source: Parikshit Gopalan and Rocco Servedio, ca. 2009.
Remarks:

- More ambitiously, one could propose the upper bound $k \cdot \binom{k-1}{\frac{k-1}{2}} 2^{1-k}$, where $k = \deg(f)$. This is achieved by the Majority function on k bits.
- Apparently, no bound better than the trivial $\sum_{i=1}^n \widehat{f}(i) \leq \mathbf{Inf}(f) \leq \deg(f)$ is known.

k-wise Independence for PTFs

Statement: Fix $d \in \mathbb{N}$ and $\epsilon \in (0,1)$. Determine the least $k = k(d, \epsilon)$ such that the following holds: If $p\colon \mathbb{R}^n \to \mathbb{R}$ is any degree-d multivariate polynomial, and $\boldsymbol{X}$ is any $\mathbb{R}^n$-valued random variable with the property that each $\boldsymbol{X}_i$ has the standard Gaussian distribution and each collection $\boldsymbol{X}_{i_1}, \dots, \boldsymbol{X}_{i_k}$ is independent, then $|\Pr[p(\boldsymbol{X}) \geq 0] - \Pr[p(\boldsymbol{Z}) \geq 0]| \leq \epsilon$, where $\boldsymbol{Z}$ has the standard n-dimensional Gaussian distribution.
Source: [DGJ+10].
Remarks:

- For $d = 1$, Diakonikolas, Gopalan, Jaiswal, Servedio, and Viola [DGJ+10] showed that $k = O(1/\epsilon^2)$ suffices. For $d = 2$, Diakonikolas, Kane, and Nelson [DKN10] showed that $k = O(1/\epsilon^8)$ suffices. For general d, Kane [Kan11b] showed that $O_d(1) \cdot \epsilon^{-2^{O(d)}}$ suffices and that $\Omega(d^2/\epsilon^2)$ is necessary.

ϵ-biased Sets for DNFs

Statement: Is it true for each constant $\delta > 0$ that $s^{-O(1)}$-biased densities δ-fool size-s DNFs? I.e., that if $f\colon \{0,1\}^n \to \{-1,1\}$ is computable by a size-s DNF and φ is an $s^{-O(1)}$-biased density on $\{0,1\}$, then $|\mathbb{E}_{\boldsymbol{x} \sim \{0,1\}^n}[f(\boldsymbol{x})] - \mathbb{E}_{\boldsymbol{y} \sim \varphi}[f(\boldsymbol{y})]| \leq \delta$.
Source: [DETT10], though the problem of pseudorandom generators for bounded-depth circuits dates back to [AW85].
Remarks:

- De, Etesami, Trevisan, and Tulsiani [DETT10] show the result for $\exp(-O(\log^2(s) \log\log s))$-biased densities. Assuming Mansour's Conjecture, their result improves to $\exp(-O(\log^2 s))$. More precisely, they show that $\exp(-O(\log^2(s/\delta) \log\log(s/\delta)))$-biased densities δ-fool size-s DNF. They also give an example showing that $s^{-O(\log(1/\delta))}$-biased densities are *necessary*. Finally, they show that $s^{-O(\log(1/\delta))}$-biased densities suffice for read-once DNFs.

PTF Sparsity for Inner Product Mod 2

Statement: Is it true that any PTF representation of the inner product mod 2 function on $2n$ bits, $\mathrm{IP}_{2n} \colon \mathbb{F}_2^{2n} \to \{-1,1\}$, requires at least 3^n monomials?
Source: Srikanth Srinivasan, 2010.
Remarks:

- Rocco Servedio independently asked if the following much stronger statement is true: Suppose $f, g \colon \{-1,1\}^n \to \{-1,1\}$ require PTFs of sparsity at least s, t, respectively; then $f \oplus g \colon \{-1,1\}^{2n} \to \{-1,1\}$ (the function $(x,y) \mapsto f(x)g(y)$) requires PTFs of sparsity at least st.

~~Servedio–Tan–Verbin Conjecture~~

Statement: Fix any $\epsilon > 0$. Then every monotone $f \colon \{-1,1\}^n \to \{-1,1\}$ is ϵ-close to a poly(deg(f))-junta.
Source: Elad Verbin (2010) and independently Rocco Servedio and Li-Yang Tan (2010)
Remarks:

- One can equivalently replace degree by decision-tree depth or maximum sensitivity.
- Disproved by Daniel Kane, 2012.

Approximate Degree for Approximate Majority

Statement: What is the least possible degree of a function $f \colon \{-1,1\}^n \to [-1,-2/3] \cup [2/3,1]$ which has $f(x) \in [2/3,1]$ whenever $\sum_{i=1}^n x_i \geq n/2$ and has $f(x) \in [-1,-2/3]$ whenever $\sum_{i=1}^n x_i \leq -n/2$?
Source: Srikanth Srinivasan, 2010.
Remarks:

- Note that $f(x)$ is still required to be in $[-1,-2/3] \cup [2/3,1]$ when $-n/2 < \sum_{i=1}^n x_i < n/2$.

Uncertainty Principle for Quadratic Fourier Analysis

Statement: Suppose $q_1, \dots, q_m \colon \mathbb{F}_2^n \to \mathbb{F}_2$ are polynomials of degree at most 2 and suppose the indicator function of $(1,\dots,1) \in \mathbb{F}_2^n$, namely $\mathrm{AND} \colon \mathbb{F}_2^n \to \{-1,1\}$, is expressible as $\mathrm{AND}(x) = \sum_{i=1}^m c_i (-1)^{q_i(x)}$ for some real numbers c_i. What is a lower bound for m?
Source: Hamed Hatami, 2011.
Remarks:

- Hatami can show that $m \geq n$ is necessary but conjectures $m \geq 2^{\Omega(n)}$ is necessary. Note that if the q_i's are of degree at most 1 then $m = 2^n$ is necessary and sufficient.
- The *Constant-Degree Hypothesis* is a similar conjecture made by Barrington, Straubing, and Thérien [BST90] in 1990 in the context of finite fields.

4 Fourier analysis on the symmetric group

Friedgut–Kalai–Naor for functions on the symmetric group

Statement: A *t-coset* is a subset of the symmetric group on n elements S_n of the form

$$\{\pi \in S_n : \pi(a_1) = b_1, \ldots, \pi(a_t) = b_t\}, \text{ where } a_i \neq a_j,\ b_i \neq b_j.$$

The subspace of $\mathbb{R}[S_n]$ spanned by the characteristic functions of the t-cosets is denoted U_t. Let $\mathcal{F} \subseteq S_n$ be a set of permutations, and let $t \in \mathbb{N}$. Denote the characteristic function of $\mathcal{F}$ by f, and let f_t be the orthogonal projection of f onto U_t. If

$$\mathbb{E}[(f - f_t)^2] \leq \varepsilon\, \mathbb{E}[f].$$

then there exists a family $\mathcal{G} \subseteq S_n$ which is a union of t-cosets such that

$$|\mathcal{F} \triangle \mathcal{G}| = O(\epsilon)|\mathcal{F}|,$$

where the implied constant does not depend on n, t.
Source: [EFF13a, EFF13c, EFF13b].
Remarks:

- Some special cases are proved in [EFF13a, EFF13c, EFF13b], with $O(\epsilon^{O(1)})$ instead of ϵ: the case $t = 1$ is treated in the regime $\mathbb{E}[f] = c/n$ for small c [EFF13a] and in the regime $\mathbb{E}[f] \in (\delta, 1-\delta)$ [EFF13c], and the general case is treated in the regime $\mathbb{E}[f] = c/n(n-1)\cdots(n-t+1)$ for small c [EFF13b].

- The question can be generalized to the subspaces U_λ spanned by the characteristic functions of the λ-cosets. Here $\lambda = \lambda_1, \ldots, \lambda_\ell$ is a partition of n and a λ-coset is

$$\{\pi \in S_n : \pi(A_1) = B_1, \ldots, \pi(A_\ell) = B_\ell\}, \text{ where } |A_i| = |B_i| = \lambda_i,\ A_i \cap A_j = \emptyset,\ B_i \cap B_j = \emptyset.$$

Edge-isoperimetry sets in the transposition graph of S_n

Notation: Consider the Cayley graph on S_n generated by all transpositions. The *boundary* ∂S of a set $A \subseteq S_n$ is the set of edges connecting A and $S_n \setminus A$. The *initial segment of the lexicographic order on S_n of size k* consists of the first k permutations in lexicographic order (when written out in full, i.e. $123, 132, 213, 231, 312, 321$).
Statement: For any $0 \leq k \leq n!$, the minimum of $|\partial A|$ on sets $A \subseteq S_n$ of size k is attained by the initial segment of the lexicographic order on S_n of size k.
Source: Conjectured by Limor Ben Efraim [Efr]; mentioned in [EFF13a, EFF13c, EFF13b].
Remarks:

- Proved for $k = c(n-1)!$ (for $0 \leq c \leq n$) by Diaconis and Shahshahani [DS81]. Related stability results in [EFF13a, EFF13c, EFF13b].

5 Real-Valued Functions

L1 total influence as a function of the degree

Statement: Let $f\colon \{0,1\}^n \to [-1,1]$ satisfy $\widehat{f}(S) = 0$ for all $|S| > d$, $S \subset \{1,\dots,n\}$. Show that

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{x\sim\{0,1\}^n} \left| \frac{f(x_1,\dots,x_n) - f(x_1,\dots,x_{i-1},1-x_i,x_{i+1},\dots,x_n)}{2} \right| \le d.$$

Source: [AA11b].
Remarks:

- Bačkurs and Bavarian [BB13] prove a bound $O(d^3 \log d)$, which can be improved to $O(d^3)$. A different argument (unpublished) gives a bound d^2.

Effect of noise on L1 norm of bounded-degree functions

Statement: Let $f\colon \{0,1\}^n \to \mathbb{R}$ satisfy $\widehat{f}(S) = 0$ for all $|S| > d$, $S \subset \{1,\dots,n\}$. Show that

$$\|T_{1-O(1/d)} f\|_1 = \Omega(\|f\|_1).$$

(That is, there exist constants c_0, c_1 such that $\|T_{1-c_0/d} f\|_1 \ge c_1 \|f\|_1$.)
Source:
Remarks:

- Oleszkiewicz showed that $\|T_{1-\epsilon} f\|_p \ge (1-\epsilon)^{\min(d^2,n)} \|f\|_p$ for $1 \le p \le \infty$.

6 Extremal and additive combinatorics

Extensions of the Ahlswede–Khachatrian theorem

Statement: Let $\mathcal{F}$ be a collection of subsets of $\{1,\dots,n\}$. Let $t \in \mathbb{Z}$, $t > 0$. We say that $\mathcal{F}$ is t-intersecting, if, for every $A_1, A_2 \in \mathcal{F}$, $|A_1 \cap A_2| \ge t$. We say that $\mathcal{F}$ is r-wise t-intersecting if, for every $A_1,\dots,A_r \in \mathcal{F}$, $|A_1 \cap \cdots \cap A_r| \ge t$. For every $p \in [0,1]$, define the product probability measure μ_p on $\{0,1\}^n$ by the formula $\mu_p(x_1,\dots,x_n) := p^{\sum_{i=1}^n x_i}(1-p)^{\sum_{i=1}^n (1-x_i)}$. Let $m(p,t)$ be the supremum of $\mu_p(\mathcal{F})$ over all t-intersecting families $\mathcal{F}$. The Ahlswede–Khachatrian theorem states that for $t \ge 2$ and $p < 1/2$,

$$m(p,t) = \max_{s\ge 0} \mu_p(\{A \subseteq [t+2s] : |A| \ge t+s\}),$$

and the maximum is attained only at $\mathcal{F}$ isomorphic to one of these families.

1. Let $m(p,t,r)$ be the supremum of $\mu_p(\mathcal{F})$ over all r-wise t-intersecting families $\mathcal{F}$ (every r sets in $\mathcal{F}$ have at least t points in common). Show that for $t \ge 2$ and $p < 1 - 1/r$,

$$m(p,t,r) = \max_{s\ge 0} \mu_p(\{A \subseteq [t+rs] : |A| \ge t+s\}),$$

and the maximum is attained only at $\mathcal{F}$ isomorphic to one of these families.

2. Let $m'(p,t)$ be the supremum of $\sqrt{\mu_p(\mathcal{F})\mu_p(\mathcal{G})}$ over all cross-t-intersecting families $\mathcal{F},\mathcal{G}$ on the same point set (every set in $\mathcal{F}$ intersects every set in $\mathcal{G}$ in at least t points). Determine whether $m'(p,t) = m(p,t)$ for $t \geq 2$ and $p < 1/2$. (Variant: replace $\sqrt{\mu_p(\mathcal{F})\mu_p(\mathcal{F})}$ with $(\mu_p(\mathcal{F}) + \mu_p(\mathcal{G}))/2$.)

3. Let $t \geq 2$ and $p < 1/2$, and suppose that $\mathcal{F}$ is a t-intersecting family satisfying $\mu_p(\mathcal{F}) \geq (1-\epsilon)m(p,t)$. Show that there exists a t-intersecting family $\mathcal{G}$ which is $O_{p,t}(\epsilon)$-close to $\mathcal{F}$ and has measure $\mu_p(\mathcal{G}) = m(p,t)$.

Source: Item (1) appears in Tokushige [Tok05].
Remarks:

- When $p < 1/(t+1)$, the optimal family is a t-junta, and so item (1) follows from the Ahlswede–Khachatrian theorem, and item (3) was proved by Friedgut [Fri08] using Fourier analysis. Analogs of items (1–3) are known in the case $t = 1$. When $p > 1 - 1/r$, $m(p,t,r) = 1$ by taking all subsets of $[n]$ of size at least $\lceil[(r-1)n + t]/r\rceil$.

Intersecting families of graphs

Statement: Let $\mathcal{F}$ be a family of graphs (a family of subsets of K_n, the complete graph on n vertices) endowed with some μ_p measure. For a collection C of graphs, we say that $\mathcal{F}$ is C-intersecting if the intersection of every two graphs in $\mathcal{F}$ contains a subgraph isomorphic to C.

1. If $\mathcal{F}$ is odd-cycle-intersecting then $\mu_p(\mathcal{F}) \leq p^3$ for all $p < 3/4$.

2. If $\mathcal{F}$ is cycle-intersecting then $\mu_p(\mathcal{F}) \leq p^3$ for all $p < 1/2$.

Source: [EFF12]
Remarks:

- Every two graphs with at least $\approx \frac{3}{4}\binom{n}{2}$ edges intersect in at least $\approx \frac{1}{4}\binom{n}{2}$ edges, so their intersection contains an odd cycle by Turán's theorem, so we need $p < 3/4$ in item (1). Similary, every two graphs with at least $\frac{1}{2}\binom{n}{2} + \frac{1}{2}n$ edges intersect in at least n edges, so their intersection contains a cycle, so we need $p < 1/2$ in item (2).

- Ellis, Filmus and Friedgut [EFF12] proved item (1) for $p \leq 1/2$. They also obtained uniqueness and stability. The Ahlswede–Khachatrian theorem implies item (2) for $p \leq 1/4$. Uniqueness and stability follow from Friedgut [Fri08] when $p < 1/4$.

Triangle removal in $\mathbb{F}_2^n$

Statement: Let $A \subseteq \mathbb{F}_2^n$. Suppose that at least $\epsilon 2^n$ elements must be removed from A in order that A be "triangle-free", that is, there are no pairs $x, y \in A$ such that $x + y \in A$. Is it true that $\Pr[x, y, x + y \in A] \geq \text{poly}(\epsilon)$?
Source: [Gre05b].
Remarks:

- Bhattacharyya and Xie [BX10] constructed an A for which the probability is at most roughly $\epsilon^{3.409}$.

- The result of Fox [Fox11], along with a reduction from directed cycle removal to triangle removal in all groups by Král, Serra, and Vena [KSV09], implies a lower bound of $1/\text{Tower}(\log(1/\epsilon))$. Here $\text{Tower}(0) := 1$, and $\text{Tower}(i) := 2^{\text{Tower}(i-1)}$.
- A Fourier-analytic proof of the above result for $\mathbb{F}_2^n$ was given by Hatami, Sachdeva, and Tulsiani [HST13].

Subspaces in Sumsets

Statement: Fix a constant $\alpha > 0$. Let $A \subseteq \mathbb{F}_2^n$ have density at least α. Is it true that $A + A$ contains a subspace of codimension $O(\sqrt{n})$?
Source: [Gre05b]
Remarks:

- The analogous problem for the group Z_N dates back to Bourgain [Bou90].
- By considering the Hamming ball $A = \{x : |x| \leq n/2 - \Theta(\sqrt{n})\}$, it is easy to show that codimension $O(\sqrt{n})$ cannot be improved. This example is essentially due to Ruzsa [Ruz99], see [Gre05b].
- The best bounds are due to Sanders [San11], who shows that $A+A$ must contain a subspace of codimension $\lceil n/(1+\log_2(\frac{1-\alpha}{1-2\alpha}))\rceil$. Thinking of α as small, this means a subspace of *dimension* roughly $\frac{\alpha}{\ln 2} \cdot n$. Thinking of $\alpha = 1/2 - \epsilon$ for ϵ small, this is codimension roughly $n/\log_2(1/\epsilon)$. In the same work Sanders also shows that if $\alpha \geq 1/2 - .001/\sqrt{n}$ then $A + A$ contains a subspace of codimension 1.
- As noted in the remarks on the Polynomial Freiman–Ruzsa/Bogolyubov Conjectures, it is also interesting to consider the relaxed problem where we only require that $A + A$ contains 99% of the points in a large subspace. Here it might be conjectured that the subspace can have codimension $O(\log(1/\alpha))$.

Polynomial Freiman–Ruzsa Conjecture (in the $\mathbb{F}_2^n$ setting)

Statement: Suppose $\emptyset \neq A \subseteq \mathbb{F}_2^n$ satisfies $|A + A| \leq C|A|$. Then A can be covered by the union of $\text{poly}(C)$ affine subspaces, each of cardinality at most $|A|$.
Source: Attributed to Marton in [Ruz99]; for the $\mathbb{F}_2^n$ version, see e.g. [Gre05a]
Remarks:

- The following conjecture is known to be equivalent: Suppose $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ satisfies $\Pr_{\boldsymbol{x},\boldsymbol{y}}[f(\boldsymbol{x})+f(\boldsymbol{y}) = f(\boldsymbol{x}+\boldsymbol{y})] \geq \epsilon$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are independent and uniform on $\mathbb{F}_2^n$. Then there exists a linear function $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that $\Pr[f(\boldsymbol{x}) = \ell(\boldsymbol{x})] \geq \text{poly}(\epsilon)$.
- The PFR Conjecture is known to follow from the **Polynomial Bogolyubov Conjecture** [GT09b]: Let $A \subseteq \mathbb{F}_2^n$ have density at least α. Then $A + A + A$ contains an affine subspace of codimension $O(\log(1/\alpha))$. One can slightly weaken the Polynomial Bogolyubov Conjecture by replacing $A + A + A$ with kA for an integer $k > 3$. It is known that any such weakening (for fixed finite k) is enough to imply the PFR Conjecture.

- Sanders [San12] has the best result in the direction of these conjectures, showing that if $A \subseteq \mathbb{F}_2^n$ has density at least α then $A + A$ contains 99% of the points in a subspace of codimension $O(\log^4(1/\alpha))$, and hence $4A$ contains all of this subspace. This suffices to give the Freiman–Ruzsa Conjecture with $2^{O(\log^4 C)}$ in place of $\mathrm{poly}(C)$.

- Green and Tao [GT09b] have proved the Polynomial Freiman–Ruzsa Conjecture in the case that A is monotone.

Routing on the hypercube

Statement: Let $S, T \subset \{0,1\}^n$ with $|S| = |T|$ and $S \cap T = \emptyset$. Let $x = (x_1, \dots, x_n) \in \{0,1\}^n$. Let $f\colon S \to T$ be a bijection such that $f(x)_i \geq x_i$ for all $i = 1, \dots, n$. Does there exist an edge-disjoint routing on the hypercube from S to T (not necessarily respecting f)? Can you do it with monotone paths?
Source: Sushant Sachdeva

7 Fourier analysis and noise sensitivity

The propeller problem

Statement: Let $n \geq 2$. Let $P_1, P_2, P_3 \subseteq \mathbb{R}^2$ be three disjoint 120 degree sectors with cusps at the origin. Let $A_1, \dots, A_{n+1} \subseteq \mathbb{R}^n$ be a partition of $\mathbb{R}^n$. That is, $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \dots, n+1\}$, $i \neq j$, and $\cup_{i=1}^{n+1} A_i = \mathbb{R}^n$. Then

$$\sum_{i=1}^{n+1} \left|\left| \int_{A_i} x d\gamma_n(x) \right|\right|_{\ell_2^n}^2 \leq \sum_{i=1}^{3} \left|\left| \int_{P_i} x d\gamma_2(x) \right|\right|_{\ell_2^2}^2 .$$

Source: [KN09, KN13].
Remarks:

- Allowing more than $n+1$ partition elements in $\mathbb{R}^n$ reduces to the above inequality.

Standard Simplex Conjecture, analytic form

Statement: Let $n \geq 2$, $\rho \in (-1, 1)$, $3 \leq k \leq n+1$. Let $\{A_1, \dots, A_k\}$ be a partition of $\mathbb{R}^n$ with $\gamma_n(A_i) = 1/k$ for all $i \in \{1, \dots, k\}$. Let $z_1, \dots, z_k \in \mathbb{R}^n$ be the vertices of a regular simplex centered at the origin. Define a partition of $\mathbb{R}^n$ so that, for all $i \in \{1, \dots, k\}$, $B_i := \{x \in \mathbb{R}^n \colon \langle x, z_i \rangle = \max_{j=1,\dots,k} \langle x, z_i \rangle\}$.

(a) If $\rho \in (0, 1)$, then

$$\sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{A_i} T_\rho 1_{A_i} d\gamma_n \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{B_i} T_\rho 1_{B_i} d\gamma_n.$$

(b) If $\rho \in (-1, 0)$, then

$$\sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{A_i} T_\rho 1_{A_i} d\gamma_n \geq \sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{B_i} T_\rho 1_{B_i} d\gamma_n.$$

Source: [IM12].
Remarks:

- For $\rho \in (-1/(k-1), 1)$, this conjecture is equivalent to the Plurality is Stablest conjecture (described below).
- The case $\rho < 0$ implies optimal UGC hardness of approximation for the MAX−k-CUT problem.
- For $k = 3$, and $0 < \rho < \rho(n)$, the conjecture holds [Hei12].
- If $k \geq 3$ and the set A_i all have positive measure but the measures are not the same, then it is known that there isn't an optimal partition that is given by a simplex [HMN14]

Standard Simplex Conjecture, probabilistic form

Statement: Fix $0 \leq \rho \leq 1$. Then among all partitions of $\mathbb{R}^n$ into $3 \leq q \leq n+1$ parts of equal Gaussian measure, the maximal noise stability at ρ occurs for a "standard simplex partition". By this it is meant a partition $A_1, \ldots, A_q$ satisfying $A_i \supseteq \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \langle a_j, x\rangle \ \forall j \neq i\}$, where $a_1, \ldots, a_q \in \mathbb{R}^n$ are unit vectors satisfying $\langle a_i, a_j\rangle = -\frac{1}{q-1}$ for all $i \neq j$. Further, for $-1 \leq \rho \leq 0$ the standard simplex partition minimizes noise stability at ρ.
Source: [IM12]
Remarks:

- Implies the Plurality Is Stablest Conjecture from [KKMO07]; in turn, the Plurality Is Stablest Conjecture implies it for $\rho \geq -\frac{1}{q-1}$.

Symmetric Gaussian Problem, analytic form

Statement: Let $\rho \in (-1, 1)$. Let $A \subseteq \mathbb{R}^n$ satisfy $-A = A$. Let $B, C \subseteq \mathbb{R}^n$ and $b, c > 0$ so that $B = \{x \in \mathbb{R}^n \colon\ ||x||_2^2 < b\}$, $C = \{x \in \mathbb{R}^n \colon\ ||x||_2^2 > c\}$, and so that $\gamma_n(A) = \gamma_n(B) = \gamma_n(C)$. Then

$$\int_{\mathbb{R}^n} 1_A T_\rho 1_A d\gamma_n \leq \max\left(\int_{\mathbb{R}^n} 1_B T_\rho 1_B d\gamma_n, \int_{\mathbb{R}^n} 1_C T_\rho 1_C d\gamma_n\right).$$

Source: [CR11, O'D12].
Remarks:

- The case where the restriction $A = -A$ is removed is well understood [MN12, Eld13].
- Even the endpoint case $\rho \to 1$ is not completely resolved [CIMW13]. Here, the statement then becomes one of minimizing Gaussian surface area among symmetric sets.
- (Krzysztof Oleszkiewicz) In the case of minimizing the Gaussian surface area of symmetric sets, it seems impossible to approximate the Gaussian measure by spheres, since already for the 3-dimensional sphere, the set of two caps is not optimal [Bar01].

Symmetric Gaussian Problem, probabilistic form

Statement: Fix $0 \leq \rho, \mu, v \leq 1$. Suppose $A, B \subseteq \mathbb{R}^n$ have Gaussian measure μ, v respectively. Further, suppose A is centrally symmetric: $A = -A$. What is the minimal possible value of $\Pr[\boldsymbol{x} \in A, \boldsymbol{y} \in B]$, when $(\boldsymbol{x}, \boldsymbol{y})$ are ρ-correlated n-dimensional Gaussians?
Source: [CR11]
Remarks:

- It is equivalent to require both $A = -A$ and $B = -B$.
- Without the symmetry requirement, the minimum occurs when A and B are opposing halfspaces; this follows from the work of Borell [Bor85].
- A reasonable conjecture is that the minimum occurs when A is a centered ball and B is the complement of a centered ball.

Grothendieck's inequality and the tiger partition

Statement: Let $n \geq 2$. Compute the following quantity, which involves the complex-time Ornstein–Uhlenbeck operator.

$$\max_{f,g\colon \mathbb{R}^n \to \{-1,1\}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(y)\sin(\langle x, y\rangle) e^{-(||x||_2^2+||y||_2^2)/2} dx dy.$$

Source: [BMMN11, Question 3.1, Question 3.2, Conjecture 5.5].
Remarks:

- For $n = 1$, the maximum is attained by $f(x) = g(x) = \text{sign}(x)$ [BMMN11, Theorem 1.3].
- In the case $n = 2$, there is a conjecture describing f, g attaining the maximum value. It is an open problem to rigorously describe such f, g.
- Finding the best f, g would lead to improved bounds on the real Grothendieck constant.

The real Grothendieck constant

Statement: Find the minimum constant K_G over all $K \in (0, \infty)$ such that, for every $m, n \in \mathbb{N}$ and for every real matrix (a_{ij}), we have

$$\max_{\{x_i\}_{i=1}^m, \{y_j\}_{j=1}^n \subseteq S^{n+m-1}} \sum_{i=1}^m \sum_{j=1}^n a_{ij} \langle x_i, y_j \rangle \leq K \max_{\{\varepsilon_i\}_{i=1}^m, \{\delta_j\}_{j=1}^n \subseteq \{-1,1\}} \sum_{i=1}^m \sum_{j=1}^n a_{ij} \varepsilon_i \delta_j.$$

Source: [BMMN11].
Remarks:

- It is known that $K_G \in (1.676, \frac{\pi}{2\log(1+\sqrt{2})})$.
- There is a corresponding conjecture for complex scalars.

The Plurality is Stablest Conjecture

Notation: Let $m \geq 2$, $k \geq 3$. For each $j \in [k]$, let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^k$ be the jth unit coordinate vector. Let $\sigma \in [k]^n$. Define $\text{PLUR}_{m,k} \colon [k]^m \to \Delta_k$ such that

$$\text{PLUR}_{m,k}(\sigma) := \begin{cases} e_j & , \text{if } |\{i \in [m] \colon \sigma_i = j\}| > |\{i \in [m] \colon \sigma_i = r\}|, \\ & \qquad \forall\, r \in [k] \setminus \{j\} \\ \frac{1}{k}\sum_{i=1}^k e_i & , \text{otherwise} \end{cases}$$

Statement: Let $n \geq 2$, $k \geq 3$, $\rho \in [-\frac{1}{k-1}, 1]$, $\varepsilon > 0$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$. Then there exists $\tau > 0$ such that, if $f \colon \{1, \ldots, k\}^n \to \Delta_k$ satisfies $\sum_{\sigma \in \{1,\ldots,k\}^n \colon \sigma_j \neq 1} (\widehat{f_i}(\sigma))^2 \leq \tau$ for all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n\}$, then

(a) If $\rho \in (0,1]$, and if $\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} f(\sigma) = \frac{1}{k}\sum_{i=1}^k e_i$, then

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}\langle f(\sigma), T_\rho f(\sigma)\rangle \le \lim_{m\to\infty}\frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m}\langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle + \varepsilon.$$

(b) If $\rho \in [-1/(k-1), 0)$, then

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}\langle f(\sigma), T_\rho f(\sigma)\rangle \ge \lim_{m\to\infty}\frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m}\langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle - \varepsilon.$$

Source: [IM12, Hei12]
Remarks:

- The case $k = 2$ was proved in [MOO10], and it is known as the Majority is Stablest Theorem. The proof used Borell's Theorem [BS01, MN12, Eld13] (a Gaussian isoperimetric result for two Euclidean sets) and an invariance principle (which relates moments of functions on the discrete hypercube $\{-1,1\}^n$ to moments of functions on Euclidean space).

Majority is Stablest under coefficient bounds

Statement: Let $f\colon \{-1,1\}^n \to \{-1,1\}$ and let $x, y \in \{-1,1\}^n$ be $1 > \rho > 0$ correlated, so that different coordinates are independent, x and y are uniformly distributed and $\mathbb{E}[x_i y_i] = \rho$ for all $i = 1,\ldots,n$ Suppose that $\mathbb{E}[f] = 0$. Is it true that

$$\mathbb{E}[f(x)f(y)] \le \frac{2}{\pi}\arcsin\rho + \epsilon_\rho(\max|\hat{f}(S)|),$$

where

$$\epsilon_\rho(\delta) \to 0 \quad \text{as} \quad \delta \to 0?$$

Source: This question was asked in the context of [MOO10] where a weaker result was proven assuming low influence instead of low coefficients.
Remarks:

- The examples of the function $n^{-1/2}(x_1 - x_2)(x_3 + \ldots x_n)$ shows that invariance does not hold assuming low coefficients [MOO10].

Majority Is Least Stable Conjecture

Statement: Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a linear threshold function, n odd. Then for all $\rho \in [0,1]$, $\mathbf{Stab}_\rho[f] \ge \mathbf{Stab}_\rho[\mathrm{Maj}_n]$.
Source: [BKS99]
Remarks:

- Slightly weaker version: If f is a linear threshold function then $\mathbf{NS}_\delta[f] \le \frac{2}{\pi}\sqrt{\delta} + o(\sqrt{\delta})$.
- The best result towards the weaker version is Peres's Theorem [Per04], which shows that every linear threshold function f satisfies $\mathbf{NS}_\delta[f] \le \sqrt{\frac{2}{\pi}}\sqrt{\delta} + O(\delta^{3/2})$.

- By taking $\rho \to 0$, the conjecture has the following consequence, which is also open: Let $f: \{-1,1\}^n \to \{-1,1\}$ be a linear threshold function with $\mathbb{E}[f] = 0$. Then $\sum_{i=1}^n \widehat{f}(i)^2 \geq \frac{2}{\pi}$. The best known lower bound here is $\frac{1}{2}$, which follows from the Khinchine–Kahane inequality; see [GL94].

Talagrand's "Convolution with a Biased Coin" Conjecture

Statement: Let $f: \{-1,1\}^n \to \mathbb{R}_{\geq 0}$ have $\mathbb{E}[f] = 1$. Fix any $0 < \rho < 1$. Then $\Pr[\mathrm{T}_\rho f \geq t] < o(1/t)$.
Source: [Tal89]
Remarks:

- Talagrand in fact suggests the bound $O(\frac{1}{t\sqrt{\log t}})$.
- Talagrand offers a \$1000 prize for proving this.
- Even the "special case" when f's domain is $\mathbb{R}^n$ with Gaussian measure is open. In this Gaussian setting, Ball, Barthe, Bednorz, Oleszkiewicz, and Wolff [BBB+13] have shown the upper bound $O(\frac{1}{t\sqrt{\log t}})$ for $n = 1$ and the bound $O(\frac{\log\log t}{t\sqrt{\log t}})$ for any fixed constant dimension.

Sensitivity versus Block Sensitivity

Statement: For any $f: \{-1,1\}^n \to \{-1,1\}$ it holds that $\deg(f) \leq \mathrm{poly}(\mathrm{sens}[f])$, where $\mathrm{sens}[f]$ is the (maximum) sensitivity, $\max_x |\{i \in [n] : f(x) \neq f(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)\}|$.
Source: [CFGS88, Sze89, GL92, NS94]
Remarks:

- As the title suggests, it is more usual to state this as $\mathrm{bs}[f] \leq \mathrm{poly}(\mathrm{sens}[f])$, where $\mathrm{bs}[f]$ is the "block sensitivity". However the version with degree is equally old, and in any case the problems are equivalent since it is known that $\mathrm{bs}[f]$ and $\deg(f)$ are polynomially related.
- The best known gap is quadratic ([CFGS88, GL92]) and it is suggested ([GL92]) that this may be the worst possible.

Gotsman–Linial Conjecture

Statement: Among degree-k polynomial threshold functions $f: \{-1,1\}^n \to \{-1,1\}$, the one with maximal total influence is the symmetric one $f(x) = \mathrm{sgn}(p(x_1 + \dots + x_n))$, where p is a degree-k univariate polynomial which alternates sign on the $k+1$ values of $x_1 + \dots + x_n$ closest to 0.
Source: [GL94]
Remarks:

- The case $k = 1$ is easy.
- Slightly weaker version: degree-k PTFs have total influence $O(k) \cdot \sqrt{n}$.
- Even weaker version: degree-k PTFs have total influence $O_k(1) \cdot \sqrt{n}$.
- The weaker versions are open even in the case $k = 2$. The $k = 2$ case may be related to the following old conjecture of Holzman: If $g: \{-1,1\}^n \to \mathbb{R}$ has degree 2 (for n even), then g has at most $\binom{n}{n/2}$ local strict minima.

- It is known that bounding total influence by $c(k) \cdot \sqrt{n}$ is equivalent to a bounding δ-noise sensitivity by $O(c(k)) \cdot \sqrt{\delta}$.
- The "Gaussian special case" was solved by Kane [Kan11a].
- The best upper bounds known are $2n^{1-1/2^k}$ and $2^{O(k)} \cdot n^{1-1/O(k)}$ [DHK+10].
- Recently, Daniel Kane [Kan13] proved that for any degree-k PTF, the maximum total influence is at most $c_k n^{1/2+o(1)}$. See [Kan13, O'D12] for a detailed history.

Mansour's Conjecture

Statement: Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be computable by a DNF of size $s > 1$ and let $\epsilon \in (0, 1/2]$. Then f's Fourier spectrum is ϵ-concentrated on a collection $\mathcal{F}$ with $|\mathcal{F}| \leq s^{O(\log(1/\epsilon))}$.
Source: [Man94]
Remarks:

- Weaker version: replacing $s^{O(\log(1/\epsilon))}$ by $s^{O_\epsilon(1)}$.
- The weak version with bound $s^{O(1/\epsilon)}$ is known to follow from the Fourier Entropy–Influence Conjecture.
- Proved for "almost all" polynomial-size DNF formulas (appropriately defined) by Klivans, Lee, and Wan [KLW10].
- Mansour [Man95] obtained the upper-bound $(s/\epsilon)^{O(\log\log(s/\epsilon)\log(1/\epsilon))}$.

Fourier Entropy–Influence Conjecture

Statement: There is a universal constant C such that for any $f\colon \{-1,1\}^n \to \{-1,1\}$ it holds that $\boldsymbol{H}[\widehat{f}^2] \leq C \cdot \mathbf{I}[f]$, where $\boldsymbol{H}[\widehat{f}^2] = \sum_S \widehat{f}(S)^2 \log_2 \frac{1}{\widehat{f}(S)^2}$ is the spectral entropy and $\mathbf{I}[f]$ is the total influence.
Source: [FK96]
Remarks:

- Proved for "almost all" polynomial-size DNF formulas (appropriately defined) by Klivans, Lee, and Wan [KLW10].
- Proved for symmetric functions and functions computable by read-once decision trees by O'Donnell, Wright, and Zhou [OWZ11].
- An explicit example showing that $C \geq 60/13$ is necessary is known. (O'Donnell, unpublished.)
- Weaker version: the "Min-Entropy–Influence Conjecture", which states that there exists S such that $\widehat{f}(S)^2 \geq 2^{-C\cdot \mathbf{I}[f]}$. This conjecture is strictly stronger than the KKL Theorem, and is implied by the KKL Theorem in the case of monotone functions.

Optimality of Majorities for Non-Interactive Correlation Distillation

Statement: Fix $r \in \mathbb{N}$, n odd, and $0 < \epsilon < 1/2$. For $f\colon \{-1,1\}^n \to \{-1,1\}$, define $P(f) = \Pr[f(\boldsymbol{y}^{(1)}) = f(\boldsymbol{y}^{(2)}) = \cdots f(\boldsymbol{y}^{(r)})]$, where $\boldsymbol{x} \sim \{-1,1\}^n$ is chosen uniformly and then each $\boldsymbol{y}^{(i)}$ is

(independently) an ϵ-noisy copy of $\boldsymbol{x}$. Is it true that $P(f)$ is maximized among odd functions f by the Majority function Maj_k on *some* odd number of inputs k?
Source: [MO05] (originally from 2002)
Remarks:

- It is possible (e.g., for $r = 10$, $n = 5$, $\epsilon = .26$) for neither the Dictator (Maj_1) nor full Majority (Maj_n) to be maximizing.

Noise Sensitivity of Intersections of Halfspaces

Statement: Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be the intersection (AND) of k linear threshold functions. Then $\mathbf{NS}_\delta[f] \le O(\sqrt{\log k}) \cdot \sqrt{\delta}$.
Source: [KOS04]
Remarks:

- The bound $O(k) \cdot \sqrt{\delta}$ follows easily from Peres's Theorem and is the best known.
- The "Gaussian special case" follows easily from the work of Nazarov [Naz03].
- An upper bound of the form $\mathrm{polylog}(k) \cdot \delta^{\Omega(1)}$ holds if the halfspaces are sufficiently "regular" [HKM12].

~~Average sensitivity of intersection of half-spaces~~

Statement: Let $f\colon \{0,1\}^n \to \{0,1\}$ be the indicator function of an intersection of k half-spaces. Then there exists a universal constant $C > 0$ such that

$$2^{-n} \sum_{x=(x_1,\ldots,x_n)\in\{0,1\}^n} |\{i \in \{1,\ldots,n\} \colon f(x) \ne f(x_1,\ldots,x_{i-1},1-x_i,x_{i+1},\ldots,x_n)\}| \le C\sqrt{n \log k}.$$

$$\mathbf{Inf}(f) \le C\sqrt{n \log k}.$$

Source: [Kan]
Remarks:

- This conjecture was settled by Daniel Kane [Kan].

Non-Interactive Correlation Distillation with Erasures

Statement: Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be an unbiased function. Let $\boldsymbol{z} \sim \{-1,0,1\}^n$ be a "random restriction" in which each coordinate $\boldsymbol{z}_i$ is (independently) ± 1 with probability $p/2$ each, and 0 with probability $1-p$. Assuming $p < 1/2$ and n odd, is it true that $\mathbb{E}_{\boldsymbol{z}}[|f(\boldsymbol{z})|]$ is maximized when f is the majority function? (Here we identify f with its multilinear expansion.)
Source: [Yan07]
Remarks:

- For $p \ge 1/2$, Yang conjectured that $\mathbb{E}_{\boldsymbol{z}}[|f(\boldsymbol{z})|]$ is maximized when f is a dictator function; this was proved by O'Donnell and Wright [OW12].
- Mossel [Mos10] shows that if f's influences are assumed at most τ then the following inequality holds: $\mathbb{E}_{\boldsymbol{z}}[|f(\boldsymbol{z})|] \le \mathbb{E}_{\boldsymbol{z}}[|\mathrm{Maj}_n(\boldsymbol{z})|] + o_\tau(1)$.

Average versus Max Sensitivity for Monotone Functions

Statement: Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be monotone. Then $\mathbf{I}[f] < o(\mathrm{sens}[f])$.
Source: Rocco Servedio, Li-Yang Tan, 2010
Remarks:

- The tightest example known has $\mathbf{I}[f] \approx \mathrm{sens}[f]^{.61}$; this appears in a work of O'Donnell and Servedio [OS07].

Noise correlation bounds

Statement: Let (Ω_i, μ_i) be pairwise independent distributions and let $\Omega = \prod_{i=1}^K \Omega_i, \mu = \prod_{i=1}^K \mu_i$, $\Omega_i = \{1,\dots,k_i\}$. Let $f_i\colon \Omega_i^n \to [-1,1]$, $f_i = \sum_{\sigma\in\{1,\dots,k_i\}^n} \widehat{f}(\sigma)W_\sigma$ satisfy

$$\|\sum_{\sigma\in\{1,\dots,k_i\}^n\colon |\sigma|\geq d} \widehat{f}(\sigma)W_\sigma\|_2^2 \leq (1-\gamma)^d.$$

Suppose further that f_1 satisfies $|\hat{f}_1(\sigma)| \leq \delta$ for all $\sigma \in \{1,\dots,k_1\}^n$ then

$$\mathop{\mathbb{E}}_{\mu^n}[f_1(x^1)\dots f_K(x^K)] \leq \epsilon(\delta), \quad \epsilon(\delta)\to 0 \quad \text{as} \quad \delta \to 0.$$

Source: Austrin and Mossel [AM13].
Remarks:

- In the special case where f_i are linear forms over the same additive group, this was shown to hold by Hamed Hatami using the Gowers–Cauchy–Schwartz inequality (see [AM13]).

KKL for continuous domains

Statement: We have $f\colon [0,1]^R \to \{1,\dots,q\}$, where $[0,1]$ is endowed with its Haar measure. We want to prove versions of [KKL88] and *majority is stablest* in this setting. The influence can be defined as

$$\mathrm{Inf}_i(f) = \mathrm{Pr}_{a,b\sim[0,1]} f(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_R) = f(x_1,\dots,x_{i-1},b,x_{i+1},\dots,x_R).$$

(This is different from the [BKK+92] definition.) We want f to be *balanced*: no value is obtained with probability more than ϵ. We want to prove that the sum of influences is at least $2-\phi(\epsilon)-o_R(1)$ (rather than 1); the Poincaré inequality gives only 1.

Now suppose that furthermore all influences are at most τ. Is it true that that the η-*noise stability* (the probability that the function changes when we flip each coordinate with probability η) is at most $1-2\epsilon$? This is tight for a discretized version of max.
Source: Prasad Raghavendra

8 Computational hardness

MAX-2-LIN(2) on the Hypercube

Statement: Suppose you have variables on the vertices of a k-hypercube. For each of the edges, we have a constraint $x_i + x_j = c$. Is it Unique Games hard to approximately satisfy $1 - \epsilon$ fraction of the constraints vs. $1 - \sqrt{\epsilon}$ fraction of the constraints? Is the corresponding question for $1 - \epsilon$ vs. $1 - M\epsilon$ easy (perhaps even using Goemans–Williamson)?
Source: [KKMO07]

Approximation Resistance of the republic function

Statement: Define the *republic* predicate as follows:

$$\mathrm{sign}(ckX_0 + \sum_{i=1}^{k} X_i).$$

Is this predicate approximation-resistant? In other words, given a set of m such constraints which is $1 - \epsilon$ satisfiable, can we find an assignment that satisfies $1/2 + \epsilon$ fraction of the constraints?
Source: Johan Håstad

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